Hybrid State Equations of Motion for Flexible Bodies in Terms of Quasi-Coordinates

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This paper is concerned with the general motion of a flexible body in space. Using the extended Hamilton's principle for distributed systems, standard Lagrange's equations for hybrid systems are first derived. Then, the equations for the rigid-body motions are transformed into a symbolic vector form of Lagrange's equations in terms of general quasi-coordinates. The hybrid Lagrange's equations of motion in terms of general quasi-coordinates are subsequently expressed in terms of quasi-coordinates representing rigid-body motions. Finally, the second-order Lagrange's equations for hybrid systems are transformed into a set of state equations suitable for control. An illustrative example is presented.

Introduction

THE derivation of the equations of motion has preoccupied dynamicists for many years, as can be concluded from texts by Whittaker,[1] Pars,[2] and Meirovitch.[3] References 1-3 consider the motion of systems of particles and rigid bodies, and the equations of motion are presented in a large variety of forms. In this paper, we concentrate on a certain formulation, namely, Lagrange's equations. For an n-degree-of-freedom system, Lagrange's equations consist of n second-order ordinary differential equations for the system displacements.

In the control of dynamical systems, it is often convenient to work with first-order rather than second-order differential equations. Introducing the velocities as auxiliary variables, it is possible to transform the n second-order equations into $2n$ first-order state equations. The state equations are widely used in modern control theory.[4]

With the advent of man-made satellites, there has been a renewed interest in the derivation of the equations of motion. The motion of rigid spacecraft can be defined in terms of translations and rotations of a reference set of axes embedded in the body and known as body axes. The equations of motion for such systems can be obtained with ease by means of Lagrange's equations. It is common practice to define the orientation of the body relative to an inertial space in terms of a set of rotations about nonorthogonal axes.[3] However, the kinetic energy has a simpler form when expressed in terms of angular velocity components about the orthogonal body axes than in terms of angular velocities about nonorthogonal axes. Moreover, for feedback control, it is more convenient to work with angular velocity components about the body axes, as sensors measure angular motions and actuators apply torques in terms of components about the body axes. In such cases, it is often advantageous to work not with standard Lagrange's equations but with Lagrange's equations in terms of quasi-coordinates.[1,3] If the body contains discrete parts, such as lumped masses connected to a main rigid body by massless springs, it is convenient to work with a set of axes embedded in the undeformed body. The equations of motion consist entirely of ordinary differential equations and can be obtained by a variety of approaches,[5] including the standard Lagrange's equations and Lagrange's equations in terms of quasi-coordinates. (Note that Ref. 5 refers to Lagrange's equations in terms of quasi-coordinates as Boltzmann-Hamel equations.)

In the more general case, the body can be regarded as being either entirely flexible with distributed mass and stiffness properties or as consisting of a main rigid body with distributed elastic appendages. Unlike the previous case, the equations of motion are hybrid, in the sense that the equations for the rigid-body motions are ordinary differential equations and those for the elastic motions are partial differential equations. Hybrid equations were obtained for the first time in Ref. 6. Moreover, the formulation of Ref. 6 was obtained by using Lagrange's equations in terms of quasi-coordinates, but some generality was lost in that the body considered was assumed to be symmetric and to undergo antisymmetric elastic motions only; the rigid-body translations were assumed to be zero.

This paper is concerned with the general motion of a flexible body in space. Using the extended Hamilton's principle for distributed systems,[7] standard Lagrange's equations for hybrid systems are first derived. Then, using the approach of Ref. 3, the equations for the rigid-body motions are transformed into a symbolic vector form of Lagrange's equations in terms of general quasi-coordinates. The hybrid Lagrange's equations of motion in terms of general quasi-coordinates are subsequently expressed in terms of quasi-coordinates representing rigid-body motions. This is a very important step, as the latter form permits the derivation of the hybrid equations of motion with relative ease. This step does not have to be repeated every time the equations of motion for a given flexible body are to be derived, so that a great deal of tedious work has been eliminated. These hybrid equations represent an extension to flexible bodies of Lagrange's differential equations in terms of quasi-coordinates derived in Ref. 3 for rigid bodies. The second-order equations are then used to derive the hybrid state equations.

As an illustration, the hybrid equations of motion of a spacecraft consisting of a rigid hub with a flexible appendage simulating an antenna are derived.

Standard Lagrange's Equations for Hybrid Systems

Let us consider a flexible body and assume that the Lagrangian $L = T - V$, in which T is the kinetic energy and V the potential energy, can be written in the general form $L = L(q_i, \dot{q}_i, u_j, \dot{u}_j, u_j', u_j'', \ldots, u_j^{(p)}, t)$, where $q_i = q_i(t)$ $(i = 1,2, \ldots, m)$ are generalized coordinates describing rigid-body motions of the body and $u_j(P,t)$ $(j = 1,2, \ldots, n)$ are distributed coordinates describing elastic motions relative to the rigid-body motions of a typical point in the body identified by the spatial position P. Dots designate derivatives with respect to

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time and primes designate derivatives with respect to the spatial position. For convenience, we express the Lagrangian in terms of the Lagrangian density $\hat{L}$ in the form $L = \int_D \hat{L}\, \mathrm{d}D$, where D is the domain of extension of the body.

We propose to derive Lagrange's equations by means of the extended Hamilton's principle,[7] which can be stated as

$$\int_{t_1}^{t_2}\int_D (\delta\hat{L} + \delta\hat{W})\, \mathrm{d}D\mathrm{d}t = 0$$

$$\delta q_i = \delta u_j = 0 \quad \text{at} \quad t = t_1, t_2 \tag{1}$$

where $\delta\hat{W}$ is the nonconservative virtual work density, which is related to the virtual work by $\delta W = \int_D \delta\hat{W}\, \mathrm{d}D$. The virtual work can be written in the form

$$\delta W = \sum_{i=1}^{m} Q_i \delta q_i + \sum_{j=1}^{n}\int_D \hat{U}_j \delta u_j\, \mathrm{d}D \tag{2}$$

where Q_i are nonconservative generalized forces associated with the rigid-body motions and $\hat{U}_j$ are nonconservative generalized force densities associated with the elastic motions; δq_i and δu_j are associated virtual displacements. Following the usual steps,[7] we obtain Lagrange's equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i, \qquad i = 1,2,\ldots,m \tag{3a}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \hat{T}}{\partial \dot{u}_j}\right) - \frac{\partial \hat{T}}{\partial u_j} + \mathcal{L}_j u_j = \hat{U}_j, \qquad j = 1,2,\ldots,n \tag{3b}$$

where $\mathcal{L}_j$ are homogeneous stiffness differential operators of order $2p$. In this regard, it should be pointed out that the stiffness operators $\mathcal{L}_j$ include all the terms that can be accounted for in a strain energy function. Of course, all these terms have been excluded from the Lagrangian density, which explains why the Lagrangian density $\hat{L}$ has been replaced in Eq. (3b) by the kinetic energy density $\hat{T}$. It turns out that, in so doing, we can maintain generality and, in some cases, simplify the derivation of the equations of motion for specific systems. The displacements u_j are subject to boundary conditions, which can be written in the form

$$B_{kj}u_j = 0 \text{ on } S, \qquad k = 1,2,\ldots,p, \quad j = 1,2,\ldots,n \tag{4}$$

where B_{kj} are homogeneous differential operators of order ranging from zero to $2p-1$ and S is the set of points bounding D. Equations (3a) are ordinary differential equations and Eqs. (3b) are partial differential equations. The equations are simultaneous because Eqs. (3a) contain u_j and Eqs. (3b) contain q_i. Because of their mixed nature, we refer to Eqs. (3) as hybrid differential equations of motion.

Equations (3) can be expressed in the symbolic vector form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q \tag{5a}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \hat{T}}{\partial \dot{u}}\right) - \frac{\partial \hat{T}}{\partial u} + \mathcal{L}u = \hat{U} \tag{5b}$$

where q and Q are m vectors, u and $\hat{U}$ are n vectors, and $\mathcal{L}$ is an $n \times n$ operator matrix.

Equations in Terms of Quasi-Coordinates for the Rigid-Body Motions

Quite often it is convenient to express the Lagrangian not in terms of the velocities $\dot{q}_i$ but in terms of linear combinations $w_\ell (\ell = 1,2,\ldots,m)$ of $\dot{q}_i$. The difference between $\dot{q}_i$ and w_ℓ is that the former represent time derivatives $\mathrm{d}q_i/\mathrm{d}t$, which can be integrated with respect to time to obtain the displacements q_i, whereas w_ℓ cannot be integrated to obtain displacements. It is customary to refer to w_ℓ as derivatives of quasi-coordinates.[3] The relation between w_ℓ and $\dot{q}_i$ can be expressed in the compact matrix form $w = A^T\dot{q}$, where the notation is obvious. Similarly, we express the velocities $\dot{q}_i$ in terms of the variables w_ℓ as $\dot{q} = Bw$, from which it follows that the $m \times m$ matrices A and B are related by $A^TB = B^TA = I$, where I is the identity matrix of order m. Our object is to derive Lagrange's equations in terms of w_ℓ instead of $\dot{q}_i$. Using the relations indicated earlier, it can be shown[3] that Eq. (5a) can be replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial w}\right) + B^TE\frac{\partial L}{\partial w} - B^T\frac{\partial L}{\partial q} = N \tag{6}$$

where the Lagrangian has the functional form $L = L(q_i, w_i, u_j, \dot{u}_j, u_j', u_j'', \ldots, u_j^{(p)}, t)$. Moreover,

$$E = \left[w^TB^T\frac{\partial a_{k\ell}}{\partial q}\right] - \left[w^TB^T\frac{\partial A}{\partial q_k}\right] \tag{7a}$$

$$N = B^TQ \tag{7b}$$

and we note that the first matrix in E is obtained by first carrying out a triple matrix product for every one of the m^2 entries in A and then arranging the resulting scalars in a square matrix. On the other hand, the second matrix in E is obtained by first generating a row matrix for every generalized coordinate q_k $(k = 1,2,\ldots,m)$ and then arranging the row matrices in a square matrix. Equation (6) represents a symbolic vector form of the Langrange equations for quasi-coordinates. The complete formulation is obtained by adjoining to Eq. (6) the equations for the elastic motion, Eq. (5b), as well as the associated boundary conditions, Eqs. (4).

General Equations in Terms of Quasi-Coordinates for a Translating and Rotating Flexible Body

Let us consider the body depicted in Fig. 1. The motion of the body can be described by attaching a set of body axes xyz to the body in the underformed state. The origin of the body axes coincides with an arbitrary point 0. Then the motion can be defined in terms of the translation of point 0 and the rotation of the body axes xyz relative to the inertial axes XYZ. The position of 0 relative to XYZ is given by the radius vector $R = [R_X, R_Y, R_Z]^T$. The rotation can be defined in terms of a set of angles θ_1, θ_2, and θ_3 (Fig. 2). Hence, the generalized coordinates are $q_1 = R_X$, $q_2 = R_Y$, $q_3 = R_Z$, $q_4 = \theta_1$, $q_5 = \theta_2$, $q_6 = \theta_3$. In addition, there are the elastic displacement components $u_x(P,t)$, $u_y(P,t)$, $u_z(P,t)$. The displacements R_X, R_Y, R_Z are measured relative to the inertial axes XYZ. On the other hand, the displacements u_x, u_y, u_z are measured relative to the body axes xyz. Moreover, the components $\dot{R}_X$, $\dot{R}_Y$, $\dot{R}_Z$ of the velocity vector $\dot{R}$ are also measured relative to XYZ. On the other hand, the angular velocity vector ω has components ω_x, ω_y, ω_z measured relative to the body axes xyz. It will prove convenient to express all motions in terms of components along the body axes. To this end, if we denote the velocity of

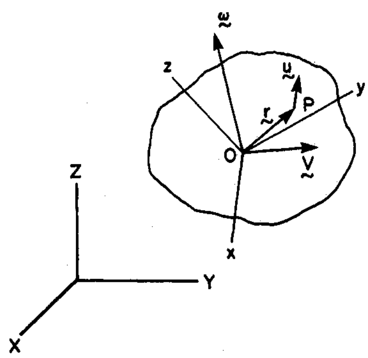

Fig. 1 Flexible body in space.

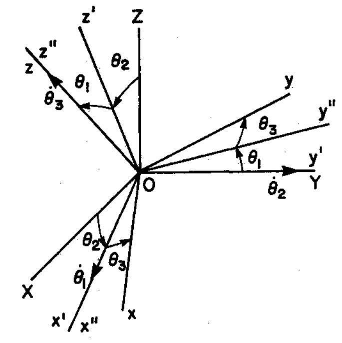

Fig. 2 Angular motions.

point 0 in terms of components along the body axes by V, then it can be shown that $V = C\dot{R}$, where $C = C(\theta_1, \theta_2, \theta_3)$ is a rotation matrix. Moreover, the angular velocity vector ω can be expressed in terms of the angular velocities $\dot\theta_1$, $\dot\theta_2$, and $\dot\theta_3$ in the form $\omega = D\dot{\theta}$, where $D = D(\theta_1, \theta_3)$ is a transformation matrix. We note that the angular velocity components ω_x, ω_y, and ω_z cannot be integrated with respect to time to yield angular displacements α_x, α_y, and α_z about axes x, y, and z, respectively. Hence, ω_x, ω_y, ω_z can be regarded as time derivatives of quasi-coordinates and treated by the procedure presented in the preceding section. Although it is not very common to regard the velocity components V_x, V_y, and V_z as time derivatives of quasi-coordinates, they can still be treated as such. In view of this, if we introduce the generalized velocity vector $\dot{q} = [\dot R_X, \dot R_Y, \dot R_Z, \dot\theta_1, \dot\theta_2, \dot\theta_3]^T$, as well as the quasivelocity vector $w = [V_x\ V_y\ V_z, \omega_x, \omega_y, \omega_z]^T$, we conclude that the coefficient matrices are defined by

$$A^T = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} \tag{8a}$$

$$B^T = A^{-1} = \begin{bmatrix} C & 0 \\ 0 & (D^T)^{-1} \end{bmatrix} \tag{8b}$$

where we recognize that $C^{-1} = C^T$ because rotation matrices are orthogonal. It can be shown, after lengthy algebraic manipulations, that

$$B^T E = \begin{bmatrix} \tilde\omega & 0 \\ \tilde V & \tilde\omega \end{bmatrix} \tag{9}$$

where $\tilde\omega$ and $\tilde V$ are the skew-symmetric matrices

$$\tilde\omega = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_z & \omega_x & 0 \end{bmatrix} \tag{10a}$$

$$\tilde V = \begin{bmatrix} 0 & -V_z & V_y \\ V_z & 0 & -V_x \\ -V_y & V_x & 0 \end{bmatrix} \tag{10b}$$

Using Eqs. (5b) and (6) in conjunction with Eqs. (8b) and (9), we obtain the hybrid Lagrange's equations in terms of quasi-coordinates

$$\frac{d}{dt}\left(\frac{\partial L}{\partial V}\right) + \tilde\omega\frac{\partial L}{\partial V} - C\frac{\partial L}{\partial R} = F \tag{11a}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \omega}\right) + \tilde V\frac{\partial L}{\partial V} + \tilde\omega\frac{\partial L}{\partial \omega} - (D^T)^{-1}\frac{\partial L}{\partial \theta} = M \tag{11b}$$

$$\frac{d}{dt}\left(\frac{\partial \hat T}{\partial v}\right) - \frac{\partial \hat T}{\partial u} + \mathfrak{L}u = \hat U \tag{11c}$$

where F and M are external nonconservative force and torque vectors, respectively, in terms of components about the body axes, $\partial L/\partial\theta = [\partial L/\partial\theta_1,\ \partial L/\partial\theta_2,\ \partial L/\partial\theta_3]^T$ and $v = \dot u$. Note that θ does not really represent a vector but a 3×1 matrix. In the next section, we relate the vectors F, M, and $\hat U$ to the actual control forces. We recall that the components of u are still subject to given boundary conditions.

It should be pointed out that, in deriving Eqs. (11), no explicit use was made of the angles θ_1, θ_2, and θ_3, so that Eqs. (11) are valid for any set of angles describing the rotation of the body axes, such as Euler's angles, and they are not restricted to the angles used here. Moreover, point 0 is an arbitrary point, not necessarily the mass center of the undeformed body, and axes xyz are not necessarily principal axes of the undeformed body. Clearly, if xyz are chosen as the principal axes with the origin at the mass center, then the equations of motion can be simplified.

State Equations in Terms of Quasi-Coordinates

Equations (11), in particular Eqs. (11a) and (11b), can be expressed in more detailed form. To this end, we write the velocity vector of a typical point P in the body in terms of components along the body axes as follows

$$v_p = V + \omega \times (r + u) + v = V + (\tilde r + \tilde u)^T\omega + v \tag{12}$$

where r is the nominal position of P relative to 0. Moreover, $\tilde r$ and $\tilde u$ represent skew-symmetric matrices associated with the vectors r and u, respectively. Then, denoting by ρ the mass density, the kinetic energy can be shown to have the expression

$$T = \frac{1}{2}\int_D \rho v_P^T v_P\, dD = \frac{1}{2}mV^TV + V^T\tilde S^T\omega + V^T\int_D \rho v\, dD + \omega^T\int_D \rho(\tilde r + \tilde u)v\, dD + \frac{1}{2}\omega^T J\omega + \frac{1}{2}\int_D \rho v^T v\, dD \tag{13}$$

where m is the total mass of the body and

$$\tilde S = \int_D \rho(\tilde r + \tilde u)\, dD \tag{14a}$$

$$J = \int_D \rho(\tilde r + \tilde u)(\tilde r + \tilde u)^T\, dD \tag{14b}$$

in which $\tilde S$ is recognized as a skew-symmetric matrix of first moments of inertia and J as a symmetric matrix of mass moments of inertia, both corresponding to the deformed body. The potential energy expression depends on the type of elastic members involved; its discussion is deferred to the next section, when an illustrative example is presented.

Inserting Eq. (13) into Eqs. (11) and rearranging, we obtain the explicit Lagrange's equations in terms of hybrid coordinates

$$m\dot V + \tilde S^T\dot\omega + \int_D \rho\dot v\, dD = (2\tilde S_v + m\tilde V + \tilde\omega\tilde S)\omega - C\frac{\partial V}{\partial R} + F \tag{15a}$$

$$\tilde S\dot V + J\dot\omega + \int_D \rho(\tilde r + \tilde u)\dot v\, dD = \left[2\int_D \rho(\tilde r + \tilde u)v\, dD + \tilde S\tilde V - \tilde\omega J\right]\omega - (D^T)^{-1}\frac{\partial V}{\partial \theta} + M \tag{15b}$$

$$\rho\dot V + \rho(\tilde r + \tilde u)^T\dot\omega + \rho\dot v = -\rho\tilde V^T\omega - \rho\tilde\omega^2(r + u) - 2\rho\tilde v^T\omega - \mathfrak{L}u + \hat U \tag{15c}$$

where u is subject to given boundary conditions. Moreover,

$$\tilde S_v = \dot{\tilde S} = \int_D \rho\tilde v\, dD \tag{16}$$

The state equations are completed by adjoining the kinematical relations

$$\dot{R} = C^T V \tag{17a}$$

$$\dot{\theta} = D^{-1}\omega \tag{17b}$$

$$\dot{u} = v \tag{17c}$$

At this point, we turn our attention to the control force and torque vectors. We denote a control force density vector at point P by $f(P,t)$ and discrete control force vectors acting at points P_i by $F_i(t)$ ($i = 1,2,\ldots,r$). Discrete forces can be treated as distributed through the use of spatial Dirac delta functions.[7] Hence, we can write the virtual work in the form

$$\delta W = \int_D \left[f^T(P,t) + \sum_{i=1}^{r} F_i^T(t)\delta(P - P_i) \right] \delta R_P \, dD \tag{18}$$

where δR_P is the virtual displacement vector of point P in terms of body-axes components. Then, by analogy with Eq. (12) for the velocity vector of P, we can write

$$\delta R_P = \delta R_{qc} + \tilde{r}^T \delta\alpha + \delta u \tag{19}$$

where δR_{qc} is the virtual displacement vector of 0, $\delta\alpha$ the virtual angular velocity vector of axes xyz, and δu the virtual elastic displacement vector, all in terms of body-axes components. Note that the term $\tilde{u}^T\delta\alpha$ was neglected as second order in magnitude. Introducing Eq. (19) into Eq. (18), we obtain

$$\delta W = \int_D \left[f^T + \sum_{i=1}^{r} F_i^T \delta(P - P_i) \right] (\delta R_{qc} + \tilde{r}^T \delta\theta + \delta u) \, dD$$

$$= F^T \delta R_{qc} + M^T \delta\alpha + \int_D \hat{U}^T \delta u \, dD \tag{20}$$

where

$$F = \int_D f \, dD + \sum_{i=1}^{r} F_i \tag{21a}$$

$$M = \int_D \tilde{r} f \, dD + \sum_{i=1}^{r} \tilde{r}_i F_i \tag{21b}$$

$$\hat{U} = f + \sum_{i=1}^{r} F_i \delta(P - P_i) \tag{21c}$$

in which r_i is the radius vector from 0 to P_i. Equations (21a) and (21b) give the resultant control force vector and resultant control torque vector acting on the body, respectively. On the other hand, Eq. (21c) gives the control force density.

Equations (15) in conjunction with Eqs. (21) are quite general and can be used readily to derive explicit equations of motion for flexible bodies of arbitrary configuration. This implies that, in deriving such explicit equations of motion, it is not necessary to duplicate the tedious steps leading to Eqs. (15) and (21). Of course, the task of determining the stiffness operator matrix $\mathcal{L}$ and the boundary conditions to be satisfied by the elastic displacement vector u remains. To carry out this task, generality must be abandoned and the configuration of the elastic body must be specified. Only then is it possible to write down the expression for the strain energy, which is a prerequisite to the derivation of $\mathcal{L}$, as demonstrated in the illustrative example that follows.

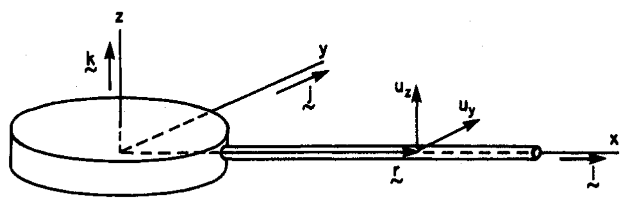

Fig. 3 Rigid spacecraft with a flexible appendage.

Illustrative Example

As an illustration, we consider a spacecraft consisting of a rigid cylindrical hub and a flexible appendage, as shown in Fig. 3. From the figure, we can write

$$r = xi, \qquad u = u_y j + u_z k, \qquad v = v_y j + v_z k \tag{22}$$

so that

$$\tilde{S} = \begin{bmatrix} 0 & -\int \rho u_z \mathrm{d}x & \int \rho u_y \mathrm{d}x \\ \int \rho u_z \mathrm{d}x & 0 & -m_1\bar{x} \\ -\int \rho u_y \mathrm{d}x & m_1 \bar{x} & 0 \end{bmatrix} \tag{23a}$$

where ρ is the mass density of the appendage, m_1 the mass of the appendage, and $\bar{x}$ the position of the mass center of the appendage. Moreover,

$$J = \begin{bmatrix} J_{xx} + \int \rho (u_y^2 + u_z^2) \mathrm{d}x & -\int \rho x u_y \mathrm{d}x & -\int \rho x u_z \mathrm{d}x \\ -\int \rho x u_y \mathrm{d}x & J_{yy} + \int \rho u_z^2 \mathrm{d}x & -\int \rho u_y u_z \mathrm{d}x \\ -\int \rho x u_z \mathrm{d}x & -\int \rho u_y u_z \mathrm{d}x & J_{zz} + \int \rho u_y^2 \mathrm{d}x \end{bmatrix} \tag{23b}$$

where J_{xx}, J_{yy}, and J_{zz} are the mass moments of inertia of the spacecraft regarded as rigid. Moreover, the matrix of direction cosines between the body axes xyz and the inertial axes XYZ for the angles θ_i ($i = 1,2,3$) used in Fig. 2 can be shown to be[3]

$$C = \begin{bmatrix} c\theta_2 c\theta_3 + s\theta_1 s\theta_2 s\theta_3 & c\theta_1 s\theta_3 & -s\theta_2 c\theta_3 + s\theta_1 c\theta_2 s\theta_3 \\ -c\theta_2 s\theta_3 + s\theta_1 s\theta_2 c\theta_3 & c\theta_1 c\theta_3 & s\theta_2 s\theta_3 + s\theta_1 c\theta_2 c\theta_3 \\ c\theta_1 s\theta_2 & -s\theta_1 & c\theta_1 c\theta_2 \end{bmatrix} \tag{24a}$$

and the transformation matrix between the angular velocity vector $\dot{\theta}$ about nonorthogonal axes and the angular velocity vector ω of the body axes is[3]

$$D = \begin{bmatrix} c\theta_3 & c\theta_1 s\theta_3 & 0 \\ -s\theta_3 & c\theta_1 c\theta_3 & 0 \\ 0 & -s\theta_1 & 1 \end{bmatrix} \tag{24b}$$

where $s\theta_i = \sin\theta_i$, $c\theta_i = \cos\theta_i$ ($i = 1,2,3$).

Using Eqs. (22–24) and ignoring second-order terms in the elastic displacements and velocities, the state equations, Eqs. (15) and (17), can be written in the explicit form

$$\dot{R}_X = (c\theta_2 c\theta_3 + s\theta_1 s\theta_2 s\theta_3) V_x - (c\theta_2 s\theta_2 - s\theta_1 s\theta_2 c\theta_3) V_y + c\theta_1 s\theta_2 V_z \tag{25a}$$

$$\dot{R}_Y = c\theta_1 s\theta_3 V_x + c\theta_1 c\theta_3 V_y - s\theta_1 V_z \tag{25b}$$

$$\dot{R}_Z = -(s\theta_2 c\theta_3 - s\theta_1 c\theta_2 s\theta_3) V_x + (s\theta_2 s\theta_3 + s\theta_1 c\theta_2 c\theta_3) V_y + c\theta_1 c\theta_2 V_z \tag{25c}$$

$$\dot{\theta}_1 = c\theta_3 \omega_x - s\theta_3 \omega_y \tag{25d}$$

$$\dot{\theta}_2 = \frac{s\theta_3}{c\theta_1}\omega_x + \frac{c\theta_3}{c\theta_1}\omega_y \tag{25e}$$

$$\dot{\theta}_3 = \frac{s\theta_1 s\theta_3}{c\theta_1}\omega_x + \frac{s\theta_1 c\theta_3}{c\theta_1}\omega_y + \omega_z \tag{25f}$$

$$\dot{u}_y = v_y \tag{25g}$$

$$\dot{u}_z = v_z \tag{25h}$$

$$m\dot{V}_x + S_{uz}\dot{\omega}_y - S_{uy}\dot{\omega}_z = mV_y\omega_z - mV_z\omega_y + m_1\bar{x}(\omega_y^2 + \omega_z^2) - S_{uy}\omega_x\omega_y - S_{uz}\omega_x\omega_z + 2S_{vy}\omega_z - 2S_{vz}\omega_y - (c\theta_2 c\theta_3 + s\theta_1 s\theta_2 s\theta_3)\frac{\partial V}{\partial R_X} - c\theta_1 s\theta_3\frac{\partial V}{\partial R_Y} + (s\theta_2 c\theta_3 - s\theta_1 c\theta_2 s\theta_3)\frac{\partial V}{\partial R_Z} + F_x \tag{25i}$$

$$m\dot{V}_y - S_{uz}\dot{\omega}_x + m_1\bar{x}\dot{\omega}_z + \int\rho\dot{v}_y\,\mathrm{d}x = mV_z\omega_x - mV_x\omega_z - m_1\bar{x}\omega_x\omega_z + S_{uy}(\omega_x^2 + \omega_z^2) - S_{uz}\omega_y\omega_z + 2S_{vz}\omega_x + (c\theta_2 s\theta_3 - s\theta_1 s\theta_2 c\theta_3)\frac{\partial V}{\partial R_X} - c\theta_1 c\theta_3\frac{\partial V}{\partial R_Y} - (s\theta_2 s\theta_3 + s\theta_1 c\theta_2 c\theta_3)\frac{\partial V}{\partial R_Z} + F_y \tag{25j}$$

$$m\dot{V}_z + S_{uy}\dot{\omega}_x - m_1\bar{x}\dot{\omega}_y + \int\rho\dot{v}_z\,\mathrm{d}x = mV_x\omega_y - mV_y\omega_x - m_1\bar{x}\omega_x\omega_z + S_{uz}(\omega_x^2 + \omega_y^2) - S_{uy}\omega_y\omega_z - S_{vy}\omega_x - c\theta_1 s\theta_2\frac{\partial V}{\partial R_X} + s\theta_1\frac{\partial V}{\partial R_Y} - c\theta_1 c\theta_2\frac{\partial V}{\partial R_Z} + F_z \tag{25k}$$

$$-S_{uz}\dot{V}_y + S_{uy}\dot{V}_z + J_{xx}\dot{\omega}_x - \bar{S}_{uy}\dot{\omega}_y - \bar{S}_{uz}\dot{\omega}_z = -(S_{uy}V_y + S_{uz}V_z)\omega_x + S_{uy}V_x\omega_y + S_{uz}V_x\omega_z + \bar{S}_{uz}\omega_x\omega_y - \bar{S}_{uy}\omega_x\omega_z + (J_{yy} - J_{zz})\omega_y\omega_z - c\theta_3\frac{\partial V}{\partial\theta_1} - \frac{s\theta_3}{c\theta_1}\frac{\partial V}{\partial\theta_2} - \frac{s\theta_1 s\theta_3}{c\theta_1}\frac{\partial V}{\partial\theta_3} + M_x \tag{25l}$$

$$S_{uz}\dot{V}_x - m_1\bar{x}\dot{V}_z - \bar{S}_{uy}\dot{\omega}_x + J_{yy}\dot{\omega}_y - \int\rho x\dot{v}_z\,\mathrm{d}x = m_1\bar{x}V_y\omega_x - (S_{uz}V_z + m_1\bar{x}V_x)\omega_y + S_{uz}V_y\omega_z - (J_{xx} - J_{zz})\omega_z\omega_x - \bar{S}_{uz}(\omega_x^2 - \omega_z^2) + \bar{S}_{uy}\omega_y\omega_z + 2\bar{S}_{vy}\omega_x + s\theta_3\frac{\partial V}{\partial\theta_1} - \frac{c\theta_3}{c\theta_1}\frac{\partial V}{\partial\theta_2} - \frac{s\theta_1 c\theta_3}{c\theta_1}\frac{\partial V}{\partial\theta_3} + M_y \tag{25m}$$

$$-S_{uy}\dot{V}_x + m_1\bar{x}\dot{V}_y - \bar{S}_{uz}\dot{\omega}_x + J_{zz}\dot{\omega}_z + \int\rho x\dot{v}_y\,\mathrm{d}x = m_1\bar{x}V_z\omega_x + S_{uy}V_z\omega_y - (S_{uy}V_y + m_1\bar{x}V_x)\omega_z + (J_{xx} - J_{yy})\omega_x\omega_y + \bar{S}_{uy}(\omega_x^2 - \omega_y^2) - \bar{S}_{uz}\omega_y\omega_z + 2\bar{S}_{vz}\omega_x - \frac{\partial V}{\partial\theta_3} + M_z \tag{25n}$$

$$\rho\dot{V}_y - \rho u_z\dot{\omega}_x + \rho x\dot{\omega}_z + \rho\dot{v}_y = \rho V_z\omega_x - \rho V_x\omega_z - \rho x\omega_x\omega_y + \rho(\omega_x^2 + \omega_z^2)u_y - \rho\omega_y\omega_z u_z + 2\rho v_z\omega_x - \mathcal{L}_z u_y + \hat{U}_y \tag{25o}$$

$$\rho\dot{V}_z + \rho u_y\dot{\omega}_x - \rho x\dot{\omega}_y + \rho\dot{v}_z = -\rho V_y\omega_x + \rho V_x\omega_y - \rho x\omega_x\omega_y - \rho\omega_y\omega_z u_y + \rho(\omega_x^2 + \omega_y^2)u_z - 2\rho v_y\omega_x - \mathcal{L}_z u_z + \hat{U}_z \tag{25p}$$

where

$$S_{uy} = \int\rho u_y\,\mathrm{d}x \tag{26a}$$

$$S_{uz} = \int\rho u_z\,\mathrm{d}x \tag{26b}$$

$$S_{vy} = \int\rho v_y\,\mathrm{d}x \tag{26c}$$

$$S_{vz} = \int\rho v_z\,\mathrm{d}x \tag{26d}$$

$$\bar{S}_{uy} = \int\rho x u_y\,\mathrm{d}x \tag{26e}$$

$$\bar{S}_{uz} = \int\rho x u_z\,\mathrm{d}x \tag{26f}$$

$$\bar{S}_{vy} = \int\rho x v_y\,\mathrm{d}x \tag{26g}$$

$$\bar{S}_{vz} = \int\rho x v_z\,\mathrm{d}x \tag{26h}$$

There remains the question of the potential energy and how it relates to the stiffness differential operators. To answer this question, we write the potential energy as

$$V = V_{el} + V_g \tag{27}$$

where V_{el} is the strain energy and V_g is the potential energy from any other sources, such as gravitational forces. Because of the nature of this paper, we focus our attention on the strain energy. We assume that the strain energy is due to the bending of the flexible appendage in the y and z directions, as well as due to the shortening of the projection,[7] where the latter is often referred to as a geometric stiffness effect. From Ref. 7, we can write the strain energy in the form

$$V_{el}(t) = \frac{1}{2}\int_h^{h+L}\{EI_y(x)[u_y''(x,t)]^2 + EI_z(x)[u_z''(x,t)]^2\}\,\mathrm{d}x + \frac{1}{2}\int_h^{h+L}\left[\int_x^L p(\zeta,t)\,\mathrm{d}\zeta\right]\{[u_y'(x,t)]^2 + [u_z'(x,t)]^2\}\,\mathrm{d}x \tag{28}$$

where h is the radius of the hub, L the length of the appendage, E the modulus of elasticity, and I_y and I_z are area moments of inertia. Moreover, $p(\zeta,t)$ is the x component of the internal force density at point ζ of the beam. This force can be obtained from the x component of Eq. (15c), which has been ignored so far. If elastic displacements are ignored because they lead to higher-order terms, the force can be verified to be

$$p(x,t) = \rho[-\dot{V}_x - \omega_y V_z + \omega_z V_y + x(\omega_y^2 + \omega_z^2)] \tag{29}$$

To generate the stiffness operators, we go back to Hamilton's principle, Eq. (1), and carry out the operations leading to Eq. (3b). More specifically, we form $\delta\hat{V}_{el}$ and evaluate the integral $\int_h^{h+L}\delta\hat{V}_{el}\,\mathrm{d}x$ by parts. In the process, we obtain not only the stiffness operators but also the boundary conditions. First, we consider Eqs. (28) and (29) and express the strain energy density as

$$\hat{V}_{el} = \frac{1}{2}[EI_y(u_y'')^2 + EI_z(u_z'')^2] + \frac{1}{2}\int_x^{h+L}\rho[-\dot{V}_x - \omega_y V_z + \omega_z V_y + \zeta(\omega_y^2 + \omega_z^2)]\,\mathrm{d}\zeta[(u_y')^2 + (u_z')^2] = \hat{V}_{el}(V_y, V_z, \omega_y, \omega_z, \dot{V}_x, u_y', u_z', u_y'', u_z'') \tag{30}$$

Consistent with the assumption that the elastic displacements are small, so that only linear terms in these displacements are to be retained in the equations of motion, we can write

$$\delta\hat{V}_{el} \cong \frac{\partial\hat{V}_{el}}{\partial u_y'}\delta u_y' + \frac{\partial\hat{V}_{el}}{\partial u_z'}\delta u_z' + \frac{\partial\hat{V}_{el}}{\partial u_y''}\delta u_y'' + \frac{\partial\hat{V}_{el}}{\partial u_z''}\delta u_z'' \tag{31}$$

Carrying out the indicated integrations by parts,[7] we obtain

$$\int_h^{h+L} \delta \hat{V}_{el}\,\mathrm{d}x = \int_h^{h+L} \left\{ \left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial \hat{V}_{el}}{\partial u_y''}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \hat{V}_{el}}{\partial u_y'}\right) \right] \delta u_y + \left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial \hat{V}_{el}}{\partial u_z''}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \hat{V}_{el}}{\partial u_z'}\right) \right] \delta u_z \right\} \mathrm{d}x + \left[\frac{\partial \hat{V}_{el}}{\partial u_y'} - \frac{\partial}{\partial x}\left(\frac{\partial \hat{V}_{el}}{\partial u_y''}\right) \right] \delta u_y \Big|_h^{h+L} + \frac{\partial \hat{V}_{el}}{\partial u_y''}\delta u_y' \Big|_h^{h+L} + \left[\frac{\partial \hat{V}_{el}}{\partial u_z'} - \frac{\partial}{\partial x}\left(\frac{\partial \hat{V}_{el}}{\partial u_z''}\right) \right] \delta u_z \Big|_h^{h+L} + \frac{\partial \hat{V}_{el}}{\partial u_z''}\delta u_z' \Big|_h^{h+L} \tag{32}$$

from which we conclude that

$$\mathcal{L}_y u_y = \frac{\partial^2}{\partial x^2}\left(\frac{\partial \hat{V}_{el}}{\partial u_y''}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \hat{V}_{el}}{\partial u_y'}\right) \tag{33a}$$

$$\mathcal{L}_z u_z = \frac{\partial^2}{\partial x^2}\left(\frac{\partial \hat{V}_{el}}{\partial u_z''}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \hat{V}_{el}}{\partial u_z'}\right) \tag{33b}$$

Hence, inserting Eq. (30) into Eqs. (33), we obtain the stiffness operators

$$\mathcal{L}_y = \frac{\partial^2}{\partial x^2}\left(EI_y \frac{\partial^2}{\partial x^2}\right) - \frac{\partial}{\partial x}\left\{ \int_x^{h+L} \rho[-\dot{V}_x - \omega_y V_z + \omega_z V_y + \zeta(\omega_y^2 + \omega_z^2)]\,\mathrm{d}\zeta \frac{\partial}{\partial x} \right\} \tag{34a}$$

$$\mathcal{L}_z = \frac{\partial^2}{\partial x^2}\left(EI_z \frac{\partial^2}{\partial x^2}\right) - \frac{\partial}{\partial x}\left\{ \int_x^{h+L} \rho[-\dot{V}_x - \omega_y V_z + \omega_z V_y + \zeta(\omega_y^2 + \omega_z^2)]\,\mathrm{d}\zeta \frac{\partial}{\partial x} \right\} \tag{34b}$$

Moreover, recognizing that

$$\int_x^{h+L} p(\zeta,t)\,\mathrm{d}\zeta = 0 \quad \text{at} \quad x = h + L \tag{35}$$

Eq. (32) in conjunction with Eq. (30) yields the boundary conditions

$$u_y = 0 \quad \text{at} \quad x = h \tag{36a}$$

$$u_y' = 0 \quad \text{at} \quad x = h \tag{36b}$$

$$EI_y \frac{\partial^2 u_y}{\partial x^2} = 0 \quad \text{at} \quad x = h + L \tag{36c}$$

$$\frac{\partial}{\partial x}\left(EI_y \frac{\partial^2 u_y}{\partial x^2}\right) = 0 \quad \text{at} \quad x = h + L \tag{36d}$$

$$u_z = 0 \quad \text{at} \quad x = h \tag{36e}$$

$$u_z' = 0 \quad \text{at} \quad x = h \tag{36f}$$

$$EI_z \frac{\partial^2 u_z}{\partial x^2} = 0 \quad \text{at} \quad x = h + L \tag{36g}$$

$$\frac{\partial}{\partial x}\left(EI_z \frac{\partial^2 u_z}{\partial x^2}\right) = 0 \quad \text{at} \quad x = h + L \tag{36h}$$

which are recognized as the boundary conditions for a cantilever beam. As a matter of interest, we identify the boundary operators in Eqs. (4) as follows

$$B_1 = 1 \quad \text{at} \quad x = h \tag{37a}$$

$$B_2 = \frac{\partial}{\partial x} \quad \text{at} \quad x = h \tag{37b}$$

$$B_1 = EI \frac{\partial^2}{\partial x^2} \quad \text{at} \quad x = h + L \tag{37c}$$

$$B_2 = \frac{\partial}{\partial x}\left(EI \frac{\partial^2}{\partial x^2}\right) \quad \text{at} \quad x = h + L \tag{37d}$$

where I is either I_y or I_z, and we note that $p = 2$ and $S = \{h, h + L\}$.

Summary and Conclusions

In deriving the equations of motion for flexible bodies by the Lagrangian approach, it is common practice to express the rotational motion in terms of angular velocities about nonorthogonal axes, which tends to complicate the equations. Moreover, this creates difficulties in feedback control in which the torque actuators apply moments about body axes and the output of sensors measuring angular motions is also expressed in terms of components about the body axes. The same can be said about force actuators and translational motion sensors. It turns out that the equations of motion are appreciably simpler when the rigid-body translations and rotations are expressed in terms of components about the body axes. Such equations can be obtained by introducing the concept of quasi-coordinates. The concept of quasi-coordinates was used earlier by this author to derive equations of motion for rotating bodies with flexible appendages, but never in the general context considered here. Indeed, in this paper, Lagrange's equations in terms of quasi-coordinates are derived for a distributed flexible body undergoing arbitrary rigid-body translations and rotations, in addition to elastic deformations. The hybrid second-order differential equations in time are then transformed into a set of hybrid state equations suitable for control design. The approach is demonstrated by deriving the hybrid state equations of motion for a spacecraft consisting of a rigid body with a flexible appendage in the form of a beam.

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